Minimal submanifolds in spheres

Abstract. Minimal submanifolds are a generalization of geodesics and soap bubbles. By definition, they are critical points of the volume functional. Given a minimal submanifold, one may ask whether it is actually a local minimum — this is the question of stability.

I review a classic result of Simons which states that minimal submanifolds of round spheres are never stable. If time permits, I will also mention some known bounds on the index, and a result by Torralbo–Urbano on what happens in (non-round) Berger spheres.

1 Preliminaries

Let M^p be a smooth manifold (possibly with boundary), $\overline{M}^n$ a Riemannian manifold with Levi-Civita connection $\overline{\nabla}$, and $f : M \to \overline{M}$ an immersion. We follow here as far as possible the conventions of Simons [5].

Definition 1. The *normal bundle* NM of M is the orthogonal complement (under the pulled back metric) of TM in $f^*T\overline{M}$, i.e. we have the orthogonal splitting

$$f^*T\overline{M} = TM \oplus NM.$$

For $X \in f^*T\overline{M}$, denote with $X^\top, X^\perp$ its projection to TM, NM, respectively. We often omit writing the pullback and treat f as if it were an inclusion, i.e. write instead $T\overline{M}\big|_M$. We may even sometimes refrain from writing the restriction and just write, for example $X^\top \in \Gamma(TM)$ for $X \in \Gamma(T\overline{M})$.

Proposition 2.
1. The Levi-Civita connection ∇ of M (with induced metric from $\overline{M}$) is given by
$$\nabla_X Y = (\overline{\nabla}_X Y)^\top, \qquad X, Y \in \Gamma(TM).$$
2. The connection ∇ on NM, defined by
$$\nabla_X W = (\overline{\nabla}_X W)^\perp, \qquad X \in \Gamma(TM),\ W \in \Gamma(NM),$$
preserves the inner product on NM.

Definition 3.
1. The *second fundamental form* A is a section of $\operatorname{Hom}(NM, \operatorname{Sym} TM)$ given by
$$A_W(X) = -(\overline{\nabla}_X W)^\top.$$
The symmetric endomorphism A_W is also called *Weingarten map/shape operator.*
2. Alternatively, one may consider the section B of $\operatorname{Sym}^2 T^*M \otimes NM$ given by
$$\langle B(X, Y), W\rangle = \langle A_W(X), Y\rangle, \qquad X, Y \in \Gamma(TM),\ W \in \Gamma(NM).$$

One may also show that

$$B(X,Y) = (\overline{\nabla}_X Y)^\perp, \qquad X, Y \in \Gamma(TM),$$

and use that as a definition.

3. The trace of B is a normal field on M called the *mean curvature* K, i.e.

$$K = \sum_{i=1}^{p} B(e_i, e_i)$$

where (e_i) is an orthonormal frame of TM.

4. Using the adjoint A^* (a section of $\mathrm{Hom}(\mathrm{Sym}\, TM, NM)$), we may define the section $\tilde{A}$ of $\mathrm{Sym}\, NM$ by

$$\tilde{A} = A^* A.$$

Lemma 4. The covariant derivative ∇B satisfies

$$\nabla_X B(Y,Z) - \nabla_Y B(X,Z) = (\bar{R}(X,Y)Z)^\perp,$$
$$\sum_{i=1}^{p} \nabla_{e_i} B(e_i, Z) = \nabla_Z K + \sum_{i=1}^{p} (\bar{R}(e_i, Z)e_i)^\perp,$$

where $X, Y, Z \in \Gamma(TM)$ and (e_i) is a local orthonormal frame of TM.

Proof. Without restriction assume that X, Y, Z, e_i are parallel with respect to ∇ at the point of calculation. Then

$$\begin{aligned}\nabla_X B(Y,Z) &= \nabla_X (B(Y,Z)) = \nabla_X (\overline{\nabla}_Y Z)^\perp = (\overline{\nabla}_X (\overline{\nabla}_Y Z)^\perp)^\perp \\ &= (\overline{\nabla}_X \overline{\nabla}_Y Z)^\perp - (\overline{\nabla}_X (\overline{\nabla}_Y Z)^\top)^\perp = (\overline{\nabla}_X \overline{\nabla}_Y Z)^\perp - B(X, \nabla_Y Z) \\ &= (\overline{\nabla}_X \overline{\nabla}_Y Z)^\perp.\end{aligned}$$

since $\nabla_Y Z = 0$ at our point. Thus

$$\begin{aligned}\nabla_X B(Y,Z) - \nabla_Y B(X,Z) &= ((\overline{\nabla}_X \overline{\nabla}_Y Z - \overline{\nabla}_Y \overline{\nabla}_X Z)^\perp \\ &= (\bar{R}(X,Y)Z)^\perp + (\overline{\nabla}_{[X,Y]} Z)^\perp = (\bar{R}(X,Y)Z)^\perp\end{aligned}$$

since $[X,Y] = \nabla_X Y - \nabla_Y X = 0$ at our point. Using the symmetry of B and the first formula, we find

$$\begin{aligned}\sum_{i=1}^{p} \nabla_{e_i} B(e_i, Z) &= \sum_{i=1}^{p} \nabla_{e_i} B(Z, e_i) = \sum_{i=1}^{p} (\nabla_Z B(e_i, e_i) + (\bar{R}(e_i, Z)e_i)^\perp) \\ &= \nabla_Z K + \sum_{i=1}^{p} (\bar{R}(e_i, Z)e_i)^\perp.\end{aligned}$$

□

2 Minimal submanifolds

Definition 5. Let $f : M \to \overline{M}$ be an immersion.

1. A *variation* of f is a 1-parameter family of immersions $f_t : M \to \overline{M}$ such that the map
$$F: \; M \times [0,1] \to \overline{M}: \quad F(p,t) = f_t(p)$$
is smooth.

2. The associated *variational vector field* $V \in \Gamma(T\overline{M}\big|_M)$ is given by
$$V_p = \frac{\partial f_t}{\partial t}\Big|_{t=0}(p) = dF\left(\frac{\partial}{\partial t}(p,0)\right).$$

Given a variation (f_t) of an immersion $f : M \to \overline{M}$ with M compact, let
$$\mathscr{A}(t) = \int_{f_t(M)} \mathrm{vol}$$
be the p-dimensional area/volume of $f_t(M)$ (the volume form is taken with respect to the induced Riemannian metric).

Theorem 6 (First variation formula)**.**
$$\mathscr{A}'(0) = -\int_M \langle V^\perp, K\rangle \,\mathrm{vol} + \int_{\partial M} *_M V^\top.$$

For small t, we may think of the first summand as measuring how strongly f_t pushes M against the direction of the mean curvature, and of the second summand as how much f_t expands M across the boundary.

Definition 7. M is called a *minimal (immersed) submanifold* if $\mathscr{A}'(0) = 0$ for all variations of f that fix the boundary, i.e. $f(\partial M) = f_t(\partial M)$ for all t. That is, minimal submanifolds are the critical points of $\mathscr{A}$.

Proposition 8. The above is is equivalent to $K \equiv 0$ (and this may also be taken as a definition of a minimal submanifold).

Example 9.

1. Geodesics are minimal submanifolds of dimension 1: they locally minimize length and are critical points of the length functional (endpoints fixed).

2. Soap films that are spanned by a fixed closed configuration of wire (the boundary) naturally arrange themselves into minimal submanifolds.

3. Standard pretty pictures include catenoids or helicoids as minimal submanifolds of $\mathbb{R}^3$. These were the first known non-flat solutions (Meunier 1776) and may of course also be realized as soap film.

4. Totally geodesic submanifolds are minimal: obviously, $A \equiv 0$ implies $K \equiv 0$.

5. An orbit of maximal area under the action of a closed group of isometries of $\overline{M}$ is always a minimal variety (with $\partial M = \emptyset$) [1].

Definition 10. For $W \in NM$ and an orthonormal frame (e_i) of TM, we define

$$\overline{\mathcal{R}}W = \sum_{i=1}^{p} (\bar{R}(e_i, W)e_i)^{\perp}.$$

$\overline{\mathcal{R}}$ is a symmetric endomorphism of NM, i.e. a section of $\operatorname{Sym} NM$.

Theorem 11 (Second variation formula)**.** Assume that (f_t) fixes the boundary. Then

$$\mathscr{A}''(0) = \int_M \langle \mathcal{J} V^{\perp}, V^{\perp} \rangle \operatorname{vol}$$

with the *Jacobi operator* $\mathcal{J} : \Gamma_0(NM) \to \Gamma_0(NM)$ given by

$$\mathcal{J} = \nabla^* \nabla + \overline{\mathcal{R}} - \tilde{A}.$$

Here $\Gamma_0(NM)$ denotes those smooth sections of NM that vanish at ∂M.

Proposition 12. The Jacobi operator is self-adjoint and strongly elliptic, hence has distinct real eigenvalues $(\lambda_i)_{i \in \mathbb{N}}$ such that

$$\lambda_1 < \lambda_2 < \ldots \to +\infty.$$

The dimension of each eigenspace is finite.

Definition 13. 1. The *index* $\operatorname{ind}(M)$ is the sum of the dimensions of eigenspaces of $\mathcal{J}$ to negative eigenvalues.

2. The *nullity* $\operatorname{nul}(M)$ is the dimension of the 0-eigenspace of $\mathcal{J}$.

3. Normal fields in the kernel of $\mathcal{J}$ are called *Jacobi fields.*

4. M is called *stable* if $\operatorname{ind}(M) = \operatorname{nul}(M) = 0$.

Proposition 14. If (f_t) is a variation of immersions (fixing the boundary) such that each (M, f_t) is a minimal immersed submanifold, then $V^{\perp}$ is a Jacobi field on M.

Proof. Suppose that (ϕ_s) is any 1-parameter family of diffeomorphisms of $\overline{M}$ that leaves $f(\partial M)$ fixed. Let

$$W = \frac{\partial \phi_s}{\partial s} \in \Gamma(T\overline{M})$$

and denote

$$v(s,t) = \mathscr{A}(\phi_s(f_t(M))).$$

For each fixed t, $f_t(M)$ is minimal and hence

$$\frac{\partial v}{\partial s}(0,t) = 0$$

(by the first variation formula applied to the variation $\phi_s \circ f_t$). Differentiating, we obtain

$$0 = \frac{\partial v}{\partial t \partial s}(0,0) = \int_M \langle \mathcal{J} V^\perp, W^\perp \rangle \,\mathrm{vol}\,.$$

Of course we can choose (ϕ_s) such that $W^\perp$ is any arbitrary element of $\Gamma_0(NM)$. Thus $\mathcal{J} V^\perp = 0$. □

The converse (does evey Jacobi field integrate into a variation?) is true for geodesics, and at least locally true in general — the proof requires hard analysis [2]. Conditions for global integrability are unknown (status 1976).

Corollary 15. If X is a Killing vector field on $\overline{M}$, then $X^\perp$ is a Jacobi field.

3 Round spheres

Take $\overline{M} = S^n$ with the round metric. The simplest example of a minimal submanifold inside S^n is S^p with the usual (totally geodesic) embedding.

Proposition 16. For this embedding, we have

$$\operatorname{ind}(S^p) = n - p, \qquad \operatorname{nul}(S^p) = (p+1)(n-p).$$

Proof. Let us first work out $\overline{\mathcal{R}}$. Since the curvature tensor of the standard sphere is given by

$$\bar{R}(X,Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y,$$

we obtain (in fact for any p-dimensional submanifold M of S^n)

$$\overline{\mathcal{R}} V = \sum_{i=1}^{p} (\bar{R}(e_i, V) e_i)^\perp = \sum_{i=1}^{p} (\langle V, e_i \rangle e_i - \langle e_i, e_i \rangle V)^\perp = -pV, \qquad V \in NM.$$

Since S^p is totally geodesic, i.e. $A \equiv 0$, the Jacobi operator thus simplifies to

$$\mathcal{J} = \nabla^* \nabla - p.$$

Taking a closer look at NS^p, one may show (by induction on the codimension) that it admits a global parallel frame $V_1, \ldots, V_{n-p}$. Writing

$$V = \sum_{i=1}^{n-p} f_i V_i, \qquad V \in \Gamma(NS^p),$$

we have (with Δ the Laplace–Beltrami operator on functions on S^p)

$$\nabla^*\nabla V = \sum_{i=1}^{n-p} (\Delta f_i) V_i.$$

Thus $\nabla^*\nabla V = \lambda V$ if and only if each of the g_i satisfies $\Delta g_i = \lambda g_i$. So

$$\ker(\nabla^*\nabla\big|_{\Gamma(NS^p)} - \lambda) \cong \bigoplus_{i=1}^{n-p} \ker(\Delta - \lambda).$$

The lowest eigenvalues of Δ are known to be 0 with multiplicity 1 and p with multiplicity $p+1$ (these are the restriction to S^p of linear functions on $\mathbb{R}^{p+1}$). In total, we see that $\mathcal{J}$ has eigenvalues $-p$ with multiplicity $n-p$, 0 with multiplicity $(p+1)(n-p)$, and all other eigenvalues are positive. □

Theorem 17. Let M^p be a compact, closed minimal immersed submanifold of S^n. Then

1. $\operatorname{ind}(M) \geq n - p$ with equality only if $M = S^p$ (totally geodesic),
2. $\operatorname{nul}(M) \geq (p+1)(n-p)$ with equality only if $M = S^p$ (totally geodesic).

To prove this theorem, we need a few auxiliary lemmata. View S^n as embedded in Euclidean $\mathbb{R}^{n+1}$, let $\overline{\overline{\nabla}}$ denote the Levi-Civita connection of $\mathbb{R}^{n+1}$, and let

$$\mathfrak{X} = \{X^{TS^n} \mid X \in \Gamma(T\mathbb{R}^{n+1}),\ \overline{\overline{\nabla}} X = 0\}$$

be the $(n+1)$-dimensional space of tangential projections to S^n of parallel vector fields on $\mathbb{R}^{n+1}$.

Lemma 18. For each $Z \in \mathfrak{X}$ there is a function $\lambda_Z \in C^\infty(S^n)$ such that

$$\overline{\nabla}_X Z = \lambda_Z X, \qquad X \in \Gamma(TS^n).$$

Proof. Let $Z = W^{TS^n}$ where W is a parallel vector field on $\mathbb{R}^{n+1}$. Then

$$\overline{\nabla}_X Z = (\overline{\overline{\nabla}}_X Z)^{TS^n} = (\overline{\overline{\nabla}}_X W^{TS^n})^{TS^n} = -(\overline{\overline{\nabla}}_X W^{NS^n})^{TS^n} = \bar{A}_{W^{NS^n}}(X)$$

where $\bar{A}$ denotes the second fundamental form of S^n in $\mathbb{R}^{n+1}$. However we know that the Weingarten map $\bar{A}_w$, $w \in N_p S^n$, is always some multiple of the identity. □

Lemma 19. For all $Z \in \mathfrak{X}$ and $X \in \Gamma(TM)$ we have

$$\begin{aligned} \nabla_X Z^\perp &= -B(X, Z^\top), \\ \nabla_X Z^\top &= A_{Z^\perp}(X) + \lambda_Z X. \end{aligned}$$

Proof. Applying Lemma 18, we find

$$\begin{aligned} \nabla_X Z^\perp &= (\overline{\nabla}_X Z^\perp)^\perp = (\overline{\nabla}_X Z - \overline{\nabla}_X Z^\top)^\perp = \lambda_Z X^\perp - (\overline{\nabla}_X Z^\top)^\perp = -B(X, Z^\top), \\ \nabla_X Z^\top &= (\overline{\nabla}_X Z^\top)^\top = (\overline{\nabla}_X Z - \overline{\nabla}_X Z^\perp)^\top = \lambda_Z X + A_{Z^\perp}(X). \end{aligned}$$

□

Lemma 20. For all $Z \in \mathfrak{X}$, we have

$$\nabla^*\nabla Z^\perp = \tilde{A} Z^\perp.$$

Proof. Let $e_1, \ldots, e_p$ be a local orthonormal frame of TM, assumed to be parallel at the point of calculation. The first formula of Lemma 19 implies that

$$\begin{aligned}\nabla^*\nabla Z^\perp &= -\sum_{i=1}^p \nabla_{e_i}\nabla_{e_i} Z^\perp = \sum_{i=1}^p \nabla_{e_i}(B(e_i, Z^\top)) \\ &= \sum_{i=1}^p (\nabla_{e_i} B(e_i, Z^\top) + B(e_i, \nabla_{e_i} Z^\top)).\end{aligned}$$

Using Lemma 4 we see that the first term is

$$\begin{aligned}\sum_{i=1}^p \nabla_{e_i} B(e_i, Z^\top) &= \nabla_{Z^\top} K + \sum_{i=1}^p (\bar{R}(e_i, Z^\top)e_i)^\perp \\ &= \nabla_{Z^\top} K + \sum_{i=1}^p (\langle Z^\top, e_i\rangle e_i^\perp - \langle e_i, e_i\rangle Z^{\top\perp}) = \nabla_{Z^\top} K.\end{aligned}$$

With the second formula of Lemma 19 it follows that

$$\nabla^*\nabla Z^\perp = \sum_{i=1}^p B(e_i, A_{Z^\perp}(e_i) + \lambda_Z e_i) = \sum_{i=1}^p B(e_i, A_{Z^\perp}(e_i)) + \lambda_Z K.$$

Since M is a minimal submanifold, $K \equiv 0$. It remains

$$\begin{aligned}\langle \nabla^*\nabla Z^\perp, W\rangle &= \sum_{i=1}^p \langle B(e_i, A_{Z^\perp}(e_i)), W\rangle = \sum_{i=1}^p \langle A_W(e_i), A_{Z^\perp}(e_i)\rangle \\ &= \langle A_W, A_Z\rangle = \langle A^*A(Z^\perp), W\rangle = \langle \tilde{A}Z^\perp, W\rangle\end{aligned}$$

for all $W \in \Gamma(NM)$, and so we see that $\nabla^*\nabla Z^\perp = \tilde{A}Z^\perp$. □

Corollary 21. For each $Z \in \mathfrak{X}$, we have

$$\left(\mathcal{J} Z^\perp, Z^\perp\right)_{L^2} = -p\|Z^\perp\|^2_{L^2}.$$

Hence $\mathcal{J}$ is negative definite on the finite-dimensional vector space $\mathfrak{X}^\perp$.

Proof. Using Lemma 20 together with $\overline{\mathcal{R}} = -p\,\mathrm{Id}$ from the proof of Proposition 16, we obtain

$$\int_M \langle \mathcal{J} Z^\perp, Z^\perp\rangle \,\mathrm{vol} = \int_M \langle \nabla^*\nabla - \overline{\mathcal{R}} Z^\perp - \tilde{A}Z^\perp, Z^\perp\rangle \,\mathrm{vol} = \int_M \langle -pZ^\perp, Z^\perp\rangle \,\mathrm{vol}\,.$$

□

Let now $\mathfrak{X}^\perp = \{Z^\perp \,|\, Z \in \mathfrak{X}\}$ denote the space of projections to NM of restrictions of vector fields in $\mathfrak{X}$.

Lemma 22. $\dim \mathfrak{X}^\perp \geq n - p$, with equality if and only if M is diffeomorphic to S^p and embedded in the standard way.

Proof. At each point of S^n, $\mathfrak{X}$ spans the entire tangent space TS^n. Thus at each point of M, $\mathfrak{X}^\perp$ spans NM. Therefore $\dim \mathfrak{X}^\perp \geq n - p$.

Suppose now that $\dim \mathfrak{X}^\perp = n - p$, and let $\mathfrak{Y}$ be the kernel of the restriction-and-projection homomorphism $\mathfrak{X} \to \mathfrak{X}^\perp$, i.e.

$$\mathfrak{Y} = \{Z \in \mathfrak{X} \,|\, Z^\top = Z\big|_M\}.$$

Fix $p \in M$ and consider the surjectiv homomorphisms

$$\alpha_p : \ \mathfrak{X} \to T_pM : \quad Z \mapsto Z_p^\top,$$
$$\beta_p : \ \mathfrak{X} \to N_pM : \quad Z \mapsto Z_p^\perp.$$

Clearly $\mathfrak{Y} \subset \ker \beta_p$. Since β_p is surjective, $\dim \ker \beta_p = n + 1 - (n - p)$. But our assumption implies that also $\dim \mathfrak{Y} = n + 1 - (n - p)$, so $\mathfrak{Y} = \ker \beta_p$. The map α_p is of course still surjective when restricted to $\ker \beta_p$, so $\alpha_p(\mathfrak{Y}) = T_pM$. This means that, given $z \in T_pM$, there always exists $Z \in \mathfrak{X}$ such that $Z_p = z$ and $Z^\perp = 0$. So for all $x, z \in T_pM$, we may apply the first part of Lemma 19 to find

$$B(x, z) = -\nabla_x Z^\perp = 0.$$

Since $p \in M$ was arbitrary, it follows that $B \equiv 0$, hence M is totally geodesic. The only such immersed submanifold of S^n is S^p. □

Corollary 21 and Lemma 22 now prove the first part of Theorem 17.

To prove the second part, let $\mathfrak{K} \subset \Gamma(TS^n)$ denote the space of Killing vector fields on S^n, and $\mathfrak{K}^\perp = \{W^\perp \,|\, W \in \mathfrak{K}\}$ the space of projections to NM of restrictions of vector fields in $\mathfrak{K}$.

Corollary 23. $\mathfrak{K}^\perp \subset \ker \mathcal{J}$.

Proof. This follows directly from Corollary 15. □

Lemma 24. For each fixed $p \in M$, $v \in N_pM$, $h \in \mathrm{Hom}(T_pM, N_pM)$, there exists $V \in \mathfrak{K}^\perp$ such that

$$V_p = v, \qquad (\nabla V)_p = h.$$

Proof. Let $\hat{h}$ be a skew-symmetric endomorphism of T_pS^n such that $\hat{h}\big|_{T_pM} = h$. By standard facts[1] about Killing vector fields on S^n, there exists a unique $W \in \mathfrak{K}$ such that

$$W_p = v, \qquad (\overline{\nabla} W)_p = \hat{h}.$$

If we set $V = W^\perp$, then $V_p = v^\perp = v$, and

$$\begin{aligned}\nabla_x V = \nabla_x W^\perp = (\overline{\nabla}_x W^\perp)^\perp &= (\overline{\nabla}_x W)^\perp - (\overline{\nabla}_x W^\top)^\perp \\ &= h(x)^\perp - B(x, v^\top) = h(x)\end{aligned}$$

for all $x \in T_pM$. □

Lemma 25. $\dim \mathfrak{K}^\perp \geq (p+1)(n-p)$, with equality if and only of M is diffeomorphic to S^n and embedded in the standard way.

Proof. Fix $p \in M$ and define

$$\varphi_p : \ \mathfrak{K}^\perp \to N_pM \oplus \mathrm{Hom}(T_pM, N_pM) : \quad V \mapsto (V_p, (\nabla V)_p).$$

By the previous lemma, φ_p is a surjective linear map. Thus

$$\dim \mathfrak{K}^\perp \geq \dim N_pM + \dim \mathrm{Hom}(T_pM, N_pM) = (n-p) + p(n-p) = (p+1)(n-p).$$

Suppose now that $\dim \mathfrak{K}^\perp = (p+1)(n-p)$. Then φ_p is an isomorphism. Thus, if $W \in \mathfrak{K}$ such that $W_p^\perp = 0$ and $(\nabla W^\perp)_p = 0$, then $W^\perp = 0$ everywhere.

Let G_p be the subgroup of $\mathrm{Isom}(S^n)$ given by

$$G_p = \{f \in \mathrm{Isom}(S^n) \,|\, f(p) = p,\ df(T_pM) = T_pM,\ df\big|_{N_pM} = \mathrm{Id}\}.$$

Then $G_p \cong \{df_p \,|\, f \in G_p\} = \mathrm{O}(T_pM)$. The Killing vector fields infinitesimally generating G_p are

$$\mathfrak{g}_p = \{W \in \mathfrak{K} \,|\, W_p = 0,\ \overline{\nabla}_x W \in T_pM \ \forall x \in T_pM,\ \overline{\nabla}_v W = 0 \ \forall v \in N_pM\}.$$

For any $W \in \mathfrak{g}_p$ we hence have $W_p^\perp = 0$ and

$$\nabla_x W^\perp = (\overline{\nabla}_x W^\perp)^\perp = (\overline{\nabla}_x W)^\perp - (\overline{\nabla}_x W^\top)^\perp = 0 - B(x, W_p^\top) = 0.$$

Thus $\varphi_p(W^\perp) = 0$. By the assumption, φ_p is an isomorphism, so we have in fact $W^\perp = 0$ everywhere, i.e. W is tangent to M. Integrating $\mathfrak{g}_p$ back to G_p, this means that G_p maps M to itself.

[1] Since $\mathrm{Isom}^0(S^n) = \mathrm{SO}(n+1)$ acts transitively on S^n with stabilizer $\mathrm{SO}(n)$, we have a Lie algebra isomorphism $\mathfrak{K} \cong \mathfrak{so}(n+1)$. Under the action of $\mathrm{SO}(n)$ we may further split $\mathfrak{so}(n+1) \cong \mathfrak{so}(n) \oplus \mathbb{R}^n$, and identifying (for fixed p) $T_pS^n \cong \mathbb{R}^n$, $\mathfrak{so}(T_pS^n) \cong \mathfrak{so}(n)$, the above isomorphism can actually be realized by

$$\mathfrak{K} \to \mathfrak{so}(n+1) \cong \mathbb{R}^n \oplus \mathfrak{so}(n) \cong T_pS^n \oplus \mathfrak{so}(T_pS^n) : \quad W \mapsto (W_p, (\overline{\nabla} W)_p).$$

This is essentially because $\overline{\nabla} W = \frac{1}{2} dW$ translates to $\mathrm{ad}(\mathrm{pr}_{\mathfrak{so}(n)} W)\big|_{\mathbb{R}^n}$ on the level of Lie algebras.

Since $G_p \cong \mathrm{O}(T_pM)$ acts transitively on the unit vectors in T_pM and holds every vector in N_pM fixed, we may conclude that $B(x,x) = B(y,y)$ for x, y unit vectors in T_pM. Thus for every orthonormal frame (e_i) of T_pM,

$$B(e_1, e_1) = \frac{1}{p} \sum_{i=1}^{p} B(e_i, e_i) = \frac{1}{p} K_p = 0,$$

hence $B \equiv 0$. Thus M is totally geodesic, and we may conclude as before that M is the standard embedded S^p. □

Corollary 23 and Lemma 25 now prove the second part of Theorem 17. □

4 Rigidity theorems

There are two interesting rigidity theorems in [5] which we state without proof. All the neighborhoods are taken with respect to a suitable topology, which we sweep under the rug here.

Theorem 26 (Extrinsic rigidity theorem)**.** Let $f : S^p \to S^n$ be the standard totally geodesic embedding. There exists a neighborhood U of f in the space of C^∞ immersions $S^p \to S^n$ such that for every minimal immersion $f' \in U$, we have $f' = g \circ f$ with $g \in \mathrm{O}(n+1)$.

Theorem 27 (Intrinsic rigidity theorem)**.** Let g denote the standard metric on S^p. Then there is a neighborhood U of g in the space of Riemannian metrics such that every $g' \in U$ not isometric to g cannot be isometrically immersed into S^n as a minimal submanifold.

However, $S^p \subset S^n$ is far from being the only minimal submanifold. Another class of examples are the Clifford tori

$$S^k\left(\sqrt{\frac{k}{n-1}}\right) \times S^{n-1-k}\left(\sqrt{\frac{n-1-k}{n-1}}\right) \subset S^n, \qquad 1 \le k \le n-2,$$

where $S^k(r) \subset \mathbb{R}^{k+1}$ denotes the sphere with radius r.

5 Index bounds for minimal hypersurfaces of the sphere

Let now $M^{n-1} \subset S^n$ be a closed minimal hypersurface. A known a-priori bound on the index is this [3, 5]:

Theorem 28. If M is not totally geodesic, then $\mathrm{ind}(M) \ge n+2$.

In fact, equality holds for the Clifford tori. If the first Betti number is known, there exists an improved bound.

Theorem 29. If $n \geq 2$ and $b_1(M) \geq 1$, then

$$\operatorname{ind}(M) \geq \frac{b_1(M)}{\binom{n+1}{2}} + n + 1.$$

Equality hold for the (unique) Clifford torus with $b_1(M) = 1$. However it is not known whether this equality characterizes the Clifford torus.

6 Berger spheres

We mention some of the results of [6] and how they contrast with the round case. First, a *Berger sphere* is $S_t^{2n+1} = (S^{2n+1}, g_t)$ with $S^{2n+1} = \{z \in \mathbb{C}^{n+1} \,|\, |z|^2 = 1\}$ and

$$g_t(v, w) = \langle v, w\rangle - (1 - t^2)\langle v, \mathrm{i}z\rangle\langle w, \mathrm{i}z\rangle, \qquad v, w \in T_z S^{2n+1}.$$

These metrics are in the canonical variation of the Hopf fibration

$$S_1 \hookrightarrow S^{2n+1} \twoheadrightarrow \mathbb{CP}^n.$$

For $0 < t < 1$, Berger spheres occur as geodesic spheres of a suitably scaled $\mathbb{CP}^{n+1}$. The isometry group is $\mathrm{U}(n+1)$.

Theorem 30. Let $f : M^d \to S_t^{2n+1}$, $0 < t < 1$, be an immersion of a closed manifold M. Then f is totally geodesic if and only if (up to congruence) it is

1. A Berger sphere S_t^{2m+1} inside $\mathbb{C}^{m+1} \subset \mathbb{C}^{n+1}$, $m < n$, or
2. $S^d = \{(x_1, \ldots, x_{d+1}, 0, \ldots, 0) \in S_t^{2n+1} \subset \mathbb{C}^{n+1} \,|\, a_i \in \mathbb{R}\}$, $1 \leq d \leq n$.

There are many examples of minimal submanifolds of S_t^{2n+1}, a lot of which may be constructed from minimal submanifolds of the base $\mathbb{CP}^n$ of the Hopf fibration. [6] computes index and nullity in many cases. We mention one of them.

Proposition 31. For the totally geodesic embedding $S_t^{2m+1} \to S_t^{2n+1}$ above, we have

$$\operatorname{ind}(S_t^{2m+1}) = \begin{cases} 0, & t^2 \leq \frac{1}{2(m+1)}, \\ 2(n-m), & \frac{1}{2(m+1)} < t^2 \leq 1, \end{cases}$$

$$\operatorname{nul}(S_t^{2m+1}) = \begin{cases} 2(n-m)(m+1), & t^2 < 1 \text{ and } t^2 \neq \frac{1}{2(m+1)}, \\ 2(n-m)(m+2), & t^2 = \frac{1}{2(m+1)}, \\ 4(n-m)(m+1), & t^2 = 1. \end{cases}$$

The most interesting fact about the Berger spheres however is that there exist *stable* minimal submanifolds for small enough t.

Theorem 32. 1. There are no stable immersed closed minimal d-dimensional submanifolds of S_t^{2n+1} when $\frac{1}{d+1} < t^2 \leq 1$.

2. If $t^2 = \frac{1}{d+1}$ for some $d \in \mathbb{N}$, then an *embedded* closed minimal submanifold M^d is stable if and only of $d = 2m+1$ and M^{2m+1} is the induced S^1-bundle via the Hopf fibration over an *embedded* complex submanifold of $\mathbb{CP}^n$.

3. If $0 < t^2 < \frac{1}{2m+2}$ for some $m \in \mathbb{N}_0$, then any such *immersed* minimal submanifold M^{2m+1} is stable.

That is, stable minimal submanifolds start to appear at $t^2 \leq \frac{1}{d+1}$, and in the case of equality at least the *embedded* ones are understood.

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