Spinors, Dirac Operators and Metric Variations

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In this article a geometric process to compare spinors for different metrics is constructed. It makes it possible to extend to spinor fields a variant of the Lie derivative (called the metric Lie derivative), giving a geometric approach to a construction originally due to Yvette Kosmann. The comparison of spinor fields for two different Riemannian metrics makes the study of the variation of Dirac operators feasible. For this, it is crucial to take into account the fact that the bundles in which the sections acted upon by the Dirac operators take their values are changing. We also give the formulas for the change in the eigenvalues of the Dirac operators. We conclude by giving a few cases in which an eigenvalue is stationary.

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Introduction

The problem we treat in this article is *how spinors and the Dirac operator depend on the metric.* Despite its importance in both physics (when gravity, i.e. the space-time metric, is coupled with other interactions) and mathematics (when the Dirac operator serves as a tool in Riemannian Geometry, starting with the fundamental formula of Lichnerowicz, cf. [16]), this question has largely been ignored in the literature.

This state of affairs may be attributed to a restrictive interpretation of the theorem with which Élie Cartan concludes the chapter *Spinor fields in Riemannian geometry* in [6] (and the book itself), and which he repeats in a note at the end of the published version of his *Notice sur les travaux scientifiques* [7]: *"The difficulties (that one faces in extending Dirac's equation from Special Relativity to General Relativity) are insurmountable if one maintains the classical technique of Riemannian Geometry: given a system of coordinates on space-time, it is impossible to represent [...] a spinor field by a finite number of components [...]"* (cf. [18] for a study of some of the physical consequences). The successful use of the Dirac operator as a model elliptic operator in novel approaches to the Index

Theorem, where reference to the metric is secondary, may be another explanation. The problem of this metric dependence has been studied in an article of E. Binz and R. Pferschy (cf. [5], see also [20]). We complete their results and give a geometric solution to the problem.

Algebraically, the notion of a *spinor* is defined only *after a a metric has been chosen*, and the difficulty in comparing spinors attached to two different metrics is connected to the following (elementary but seldomly explicitly stated) algebraic fact: *the finite-dimensional representations of the spin group* $\mathrm{Spin}(n)$ *(a two-fold covering of the special orthogonal group* $\mathrm{SO}(n)$*) which are spinorial cannot be extended to representations of the two-fold cover* $\widetilde{\mathrm{GL}}(n)$ *of the general linear group* $\mathrm{GL}(n)$, because all finite-dimensional representations of $\widetilde{\mathrm{GL}}(n)$ are themselves already representations of $\mathrm{GL}(n)$ (cf. [21]).

While defining geometric objects only up to isomorphism is unproblematic in the algebraic setting, the same cannot be said after attaching "parameters" to the theory, that is, when one takes sections of bundles of these objects and wants to differentiate them.

This is precisely the case for spinor fields on which the Dirac operator is naturally defined. The formalism we develop for the analysis of the metric dependence has the advantage of being geometric, allowing us to evaluate *variations of the Dirac operator* and subsequently *the dependence of its eigenvalues on the metric.*

The article is organised as follows: in Sec. 1 we establish the natural equivalence between any two Euclidean structures on a vector space, on which the entire article rests. This construction enables us to define a notion of metric Lie derivative, which Sec. 2 is devoted to. Above all, it makes it possible to compare spinors for different metrics, further allowing us to extend the notion of metric Lie derivative to these fields. Such an extension had already been introduced by Yvette Kosmann in [14]. We then give in Sec. 3 the explicit formulas describing the variation of the Dirac operator and its eigenvalues.

The reader unfamiliar with the notions of spinors and Dirac operator may find it useful to refer to [12] and to the recent book [15].

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1 Natural equivalence between Euclidean structures on a vector space

Let V be an n-dimensional real vector space, and denote by $\mathscr{M}V$ its cone of Euclidean metrics. It is well-known that any two such metrics g and h can always be transformed into one another by a linear mapping l (i.e. one for which one has $h = l^*g$).

We shall concern ourselves with the problem of making the definition of l *canonical*, an indispensable ingredient for a "parameter-dependent" theory such as that of Riemannian metrics on a manifold. For this later geometric use, it is helpful to formulate this correspondence as follows.

We denote with $\mathscr{B}V$ the space of *bases of* V, viewed in the spirit of the theory of principal fibre bundles as an open subset of the space $L(\mathbb{R}^n, V)$ of linear maps from $\mathbb{R}^n$ to V, by identifying an element f of this open subset with the basis $(f(e_i))$ (where (e_i) is the standard basis of $\mathbb{R}^n$). The group $\mathrm{GL}(n, \mathbb{R})$ of linear automorphisms of $\mathbb{R}^n$ acts transitively and freely from the right on $\mathscr{B}V$, so that $\mathscr{B}V$ can be identified with a copy of $\mathrm{GL}(n,\mathbb{R})$ in which there is "no distinguished (identity) element". (The group $\mathrm{GL}(V)$ of linear automorphisms of V also acts transitively on $\mathscr{B}V$, although from the left.) Furthermore, if g is a Euclidean metric on V, we denote by $\mathscr{B}_gV$ the space of *g-orthonormal bases of V*. The orthogonal group $\mathrm{O}(n)$ acts on $\mathscr{B}_gV$ transitively and freely from the right, making $\mathscr{B}_gV$ a submanifold of $\mathscr{B}V$ (and the orthogonal group of V with respect to g, denoted O_gV, acts again from the left). It is worth noting that, if $f \in \mathscr{B}_gV$ and if e denotes the standard Euclidean metric on $\mathbb{R}^n$, then

$$g = (f^{-1})^*e.$$

For any pair of metrics $g, h \in \mathscr{M}V$, we generically denote by $H_g = g^{-1} \cdot h$ the g-symmetric automorphism of V associated to h via the duality defined by g, that is, for $u, v \in V$ by

$$h(u,v) = g(H_g(u), v),$$

We thus have

1.1 Proposition (cf. [5]). *Let g and h be two Euclidean metrics on a vector space V. The linear map b^g_h, defined on a g-orthonormal basis f as*

$$b^g_h(f) = H_g^{-1/2} \circ f$$

(where, for a positive-definite symmetric endomorphism A, $A^{1/2}$ denotes the positive-definite square root), is a diffeomorphism from $\mathscr{B}_gV$ to $\mathscr{B}_hV$ which is natural in the following sense:

(i) $b^g_h = (b^h_g)^{-1}$,

(ii) b^h_g *commutes with the right action of* $\mathrm{O}(n)$ *on* $\mathscr{B}V$,

(iii) *for any differentiable curve of metrics $t \mapsto g_t$ on V, $b^g_{g_t}$ is an isotopy from $\mathscr{B}_gV$ to $\mathscr{B}_{g_t}V$.*

Proof. We first verify the algebraic properties. If f is g-orthonormal, i.e. $g = (f^{-1})^*e$, then

$$((b^g_h(f))^{-1})^*e = (H_g^{1/2})^*(f^{-1})^*e = (H_g^{1/2})^*g = h,$$

since for $v \in V$

$$((H_g^{1/2})^*g)(v,v) = g(H_g^{1/2}(v), H_g^{1/2}(v)) = h(v,v).$$

This shows that $b^g_h(f)$ is in fact h-orthonormal.

Taking the map $G_h = h^{-1} \cdot g$ and $v, w \in V$, we have

$$h(G_h(v), w) = g(v,w) = g(H_g \circ (H_g)^{-1}(v), w) = h(H_g^{-1}(v), w),$$

showing that $G_h^{-1} = H_g$ and thus $b^g_h = (b^h_g)^{-1}$.

For (ii), if $U \in \mathrm{O}(n)$, we have

$$b^g_h(f \circ U) = H_g^{-1/2} \circ f \circ U = (H_g^{-1/2} \circ f) \circ U = b^g_h(f) \circ U.$$

The end of the proof is now simply a consequence of the fact that the map $S \mapsto S^{-1/2}$ is a diffeomorphism from the cone of symmetric positive-definite endomorphisms to itself.

Lastly, it suffices to see that, g being fixed, the map $h \mapsto H_g$ is differentiable, which is evident. □

1.2 Remark. (i) In general, if l_1 and l_2 are two linear automorphisms of V, one has

$$b^g_{(l_2\circ l_1)^*g} \neq b^{l_2^*g}_{(l_2\circ l_1)^*g} \circ b^g_{l_2^*g}\ ,$$

since, if we set $h = l_2^*g$ and $k = (l_2 \circ l_1)^*g$, then $K_g^{-1/2} \neq K_h^{-1/2} \circ H_g^{-1/2}$ unless certain commutation properties are verified to which we shall return later.

(ii) If $h = a^2 g$ for some $a \in \mathbb{R}^+$, the expression reduces to $b^g_h = a^{-1}\mathrm{Id}$.

The geometric content of the map b^g_h is not entirely clear from Proposition 1.1, but shall be illuminated by Proposition 1.3 which gives an alternative definition. The following discussion is a variation on the *Polar Decomposition Theorem* which states that $\mathrm{GL}(n,\mathbb{R})$ is diffeomorphic to the product of $\mathrm{O}(n)$ which the cone $\mathscr{C}_n$ of symmetric positive-definite metrics, realised by the decomposition of any matrix A into an orthogonal matrix U and a symmetric positive-definite matrix $S = (A^\top A)^{1/2}$.

Consider the projection $p : \mathscr{B}V \to \mathscr{M}V$ defined by

$$p(f) = (f^{-1})^* e.$$

The Polar Decomposition implies in particular that the fibration p is trivial. Its fibre over g is precisely $\mathscr{B}_gV$. Identifying the tangent space $T_f\mathscr{B}V$ with $L(\mathbb{R}^n, V)$, let $\mathscr{A}_fV$ and $\mathscr{S}_fV$ be the subspaces of $L(\mathbb{R}^n, V)$ consisting of maps whose matrices with respect to f are *antisymmetric* and, respectively, *symmetric*. The subspace $\mathscr{A}_fV$ is then precisely the *vertical* space of the fibration p at f, and $\mathscr{S}_fV$ is a natural choice for a complement. The distribution $f \mapsto \mathscr{S}_fV$ is clearly $\mathrm{O}(n)$-equivariant (since the conjugate of a symmetric matrix by an orthogonal matrix is again symmetric), so it may be considered as the horizontal distribution of an $\mathrm{O}(n)$-connection which we christen the *natural connection* on the fibre bundle p.

Let us recall a few well-known facts. The space $\mathscr{M}V$, which one may identify with the symmetric space $\mathrm{GL}(n,\mathbb{R})/\mathrm{O}(n)$, carries a natural Riemannian metric $\langle\cdot,\cdot\rangle$ which is given at a point g by

$$\langle k, k'\rangle_g = \mathrm{tr}(K_g \circ K'_g).$$

This metric is complete and has non-positive curvature: two elements h and h' in $\mathscr{M}V$ are contained inside a common maximal flat subspace through g if and only if the associated endomorphisms H_g and H'_g commute.

Among the curves joining g with h that are contained in a flat subspace, we find first the geodesic from g to h with respect to the metric $\langle\cdot,\cdot\rangle$ which may be written as

$$t \mapsto g_t = g \cdot H^t_g$$

(where H^t_g denotes the symmetric positive-definite t-th power of H_g), and second the line segment

$$t \mapsto (1-t)g + th.$$

We thus have

1.3 Proposition. *Let $g, h \in \mathscr{M}V$. The transformation b^g_h coincides with the parallel transport in $\mathscr{B}V$ with respect to the natural connection along any curve joining g with h that is contained in a flat subspace of $\mathscr{M}V$.*

Proof. The parallel transport $\mathscr{B}_g V \to \mathscr{B}_h V$ along any curve $t \mapsto g_t$ joining g with h may be obtained by integrating in $\mathscr{B}V$ the differential equation

$$\frac{\mathrm{d}f_t}{\mathrm{d}t} = -\frac{1}{2}\left(\frac{\mathrm{d}G_t}{\mathrm{d}t}\right)_{g_t} \circ f_t.$$

The image of $f_0 \in \mathscr{B}_g V$ is then the solution at time 1 of the above equation with f_0 as initial condition.

We determine this horizontal lift in the case of the line segment between g and h, where $\dot{g}_t = h - g$ is constant. The differential equation can thus be rewritten as

$$\frac{\mathrm{d}f_t}{\mathrm{d}t} \circ f_t^{-1} = -\frac{1}{2}(H_g - I) \circ (I + t(H_g - I))^{-1}$$

which admits the solution $f_t = (I + t(H_g - I))^{-1/2} \circ f_0$ for $f_0 \in \mathscr{B}_g V$. We thus arrive at

$$f_1 = H_g^{-1/2} \circ f_0 = b_h^g(f_0).$$

In order to show that for any curve σ joining g with h lying entirely inside a flat subspace of $\mathscr{M}V$ containing g and h, the parallel transport along σ depends only on g and h, we need the following lemma which determines the curvature of the natural connection on the fibre bundle $p : \mathscr{B}V \to \mathscr{M}V$.

1.4 Lemma. *Let $k, k' \in T_g\mathscr{M}V$. The curvature of the natural connection on the fibre bundle $p : \mathscr{B}V \to \mathscr{M}V$ is given by*

$$\Omega_{k,k'}(f) = -\frac{1}{4}[K_g, K'_g] \circ f\,.$$

Proof. The left hand side is given by the vertical part (using the projection defined by the connection) of the Lie bracket of the horizontal lifts of two vector fields extending k and k' defined in a neighbourhood of g. Since $\mathscr{M}V$ is an open subset of the vector space $\mathrm{Sym}^2 V^*$, we take as extensions of k and k' the corresponding constant vector fields which we again denote by k and k'. The horizontal lift of $k \in T_g\mathscr{M}V$ at a g-orthonormal frame f is given by $-\frac{1}{2}K_g \circ f \in T_f\mathscr{B}V$. As we have already noted, the integral curve of the horizontal lift of the constant vector field k starting in a frame $f_0 \in \mathscr{B}_g V$ is

$$f_t = (I + tK_g)^{-1/2} \circ f_0\,.$$

The Lie bracket we seek may be calculated as the second order term in the expression for the endpoint of the curve obtained by successively traversing the integral curves of the horizontal lifts of k and k' for the times t, s, $-t$ and $-s$. This expression is

$$(I - sK'_{g+sk'})^{-1/2} \circ (I - tK_{g+tk+sk'})^{-1/2} \circ (I + sK'_{g+tk})^{-1/2} \circ (I + tK_g)^{-1/2} \circ f_0\,,$$

whose st term is easily calculated with the help of the identities

$$\begin{aligned} K'_{g+tk} &= (I + tK_g)^{-1} \circ K'_g\,, \\ K_{g+tk+sk'} &= (I + tK_{g+sk'})^{-1} \circ (I + sK'_g)^{-1} \circ K_g\,, \\ K'_{g+sk'} &= (I + sK'_g)^{-1} \circ K'_g\,. \end{aligned}$$

Keeping only the first order terms and those with st in the Taylor expansion, one obtains

$$
\begin{aligned}
(I - sK'_{g+sk'})^{-1/2} &\sim I + \frac{1}{2}sK'_g\,,\\
(I - tK_{g+tk+sk'})^{-1/2} &\sim I + \frac{1}{2}tK_g - \frac{1}{2}stK'_g \circ K_g\,,\\
(I + sK'_{g+tk})^{-1/2} &\sim I - \frac{1}{2}sK'_g + \frac{1}{2}stK_g \circ K'_g\,,\\
(I + tK_g)^{-1/2} &\sim I - \frac{1}{2}tK_g\,.
\end{aligned}
$$

Putting everything together, we arrive at the final result $\Omega_{k,k'}(f) = -\frac{1}{4}[K_g, K'_g] \circ f_0$ (note that this is already vertical since $[K_g, K'_g]$ is antisymmetric). □

Proof of Proposition 1.3 – Continued. We now find ourselves in a familiar situation. Any two curves connecting g with h which both lie in a flat subspace of $\mathscr{M}V$ containing g and h can be viewed as the trajectories (i.e. integral curves) between times 0 and 1 of vector fields on $\mathscr{M}V$. The bracket of the horizontal lifts in $\mathscr{B}V$ of these vector fields coincides with the lift of the bracket downstairs, and their flow satisfies the same relations as that of their projections to $\mathscr{M}V$. Their trajectories will therefore also meet at time 1. □

1.5 Corollary. *Let $g, h, k \in \mathscr{M}V$ be three metrics inside a common flat subspace of $\mathscr{M}V$. Then $b^g_k = b^h_k \circ b^g_h$.*

Proof. It suffices to take a curve joining g with k by ways of h and to apply Proposition 1.3.

One also directly verifies that, if $[H_g, K_g] = 0$, we have[1]

$$H_k^{-1/2} \circ K_g^{-1/2} = (H_k \circ K_g)^{-1/2}.$$

□

Proposition 1.3, being a geometric version of Proposition 1.1, can directly be extended to the spinorial setting. In order to do so, let us choose a realisation $\widetilde{\mathscr{B}}V$ of the universal (two-fold) cover of $\mathscr{B}V$. Each fibre $\mathscr{B}_gV$ is then non trivially covered by a manifold $\widetilde{\mathscr{B}}_gV$ diffeomorphic to the group $\mathrm{Pin}(n)$ which we shall call the space of *spinorial bases of* V (relative to g and the covering $\widetilde{\mathscr{B}}V$).

1.6 Proposition. *The natural map b which to any pair of metrics g and h associates the diffeomorphism $b^g_h : \mathscr{B}_gV \to \mathscr{B}_hV$ lifts to a natural map β on the space of spinorial bases $\widetilde{\mathscr{B}}V$, associating to any pair of metrics g and h a* $\mathrm{Pin}(n)$*-equivariant diffeomorphism $\beta^g_h : \widetilde{\mathscr{B}}_gV \to \widetilde{\mathscr{B}}_hV$.*

Proof. The horizontal distribution $f \mapsto \mathscr{S}_fV$ naturally lifts to a horizontal distribution in $\widetilde{\mathscr{B}}V$, determining a $\mathrm{Pin}(n)$-equivariant connection on the bundle $\widetilde{\mathscr{B}}V$, which we view as a $\mathrm{Pin}(n)$-principal bundle over $\mathscr{M}V$.

The map β^g_h is now *defined* as the parallel transport in the principal bundle $\widetilde{\mathscr{B}}V$ along any curve joining g with h that is contained in a flat subspace. □

[1] In the original article, the formula reads $H_k^{-1/2} \circ K_g^{-1/2} = (H_g \circ K_g)^{-1/2}$.

1.7 Remark. The previous considerations cannot be extended as they stand to the case of Lorentzian metrics l on an $(n+1)$-dimensional vector space V, for reasons we shall now state.

Take the Minkowski metric

$$m = (\varepsilon^1)^2 + \dots (\varepsilon^n)^n - (\varepsilon^{n+1})^2$$

as a model metric on $\mathbb{R}^{n+1}$ (where we denote by (ε^i) the dual of the standard basis). The map

$$q(f) = (f^{-1})^* m$$

defines again a fibration $q : \mathscr{B}V \to \mathscr{L}V$, where $\mathscr{L}V$ denotes the space of Lorentzian metrics on V, an open (non-convex!) cone in $\mathrm{Sym}^2 V^*$. The vertical subspace at $f \in \mathscr{B}V$ is then naturally identified with the set of maps $f \circ B : \mathbb{R}^{n+1} \to V$ where B is any matrix in the Lie algebra of the Lorentz group $\mathrm{O}(n,1)$.

Again, there is an $\mathrm{O}(n,1)$-invariant horizontal distribution, given by elements which are symmetric in the Lorentzian sense, determining a parallel transport along any curve in $\mathscr{L}V$. It is, however, generally not possible to distinguish a privileged family of curves joining two given elements of $\mathscr{L}V$. In particular, linear interpolation does not allow us to join two elements that are not already joined by a geodesic of the natural metric $\langle\cdot,\cdot\rangle$ which is again given by $\langle k, k'\rangle_l = \mathrm{Trace}\,(K_l \circ K'_l)$ (note that this metric is not Riemannian, but of signature $(\frac{1}{2}(n^2+n+2), n)$).

Moreover, it is not clear how to give a formula for the natural map between two Lorentzian metrics l and k since, if the automorphism $K_l = l^{-1} \cdot k$ is well-defined, it does not necessarily have only positive eigenvalues. It is thus not possible to take a square root without choosing how to cut the complex plane, creating difficulties in the parametric case which is after all our aim.

However, these difficulties disappear when one stays merely in a neighbourhood of a given Lorentzian metric, which is totally sufficient for extending the construction of a metric Lie derivative, given in Sec. 2, to the Lorentzian case.

2 Metric Lie derivative of tensor and spinor fields

The usual Lie derivative, which we denote by $\mathscr{L}_X$, is based on the possibility of transporting tensors along the flow (ξ_t) of any vector field X. Since this is not possible for spinor fields, we shall replace it, via the transformations $b^g_{\xi_t^* g}$ and $\beta^g_{\xi_t^* g}$ introduced in the preceding paragraphs, by a *metric Lie derivative* $\mathscr{L}^g_X$ acting on tensor as well as spinor fields.

Like the usual Lie derivative, $\mathscr{L}^g_X$ is a derivation with respect to the tensor product. However, unlike the usual derivative, it preserves the metric g, i.e. for any vector field X we have $\mathscr{L}^g_X g = 0$ (see Proposition 2.9), but also has *curvature* in the sense that $\mathscr{L}^g_{[X,Y]}$ is in general different from the commutator $[\mathscr{L}^g_X, \mathscr{L}^g_Y]$ (see Proposition 2.12). The two derivations $\mathscr{L}_X$ and $\mathscr{L}^g_X$ on tensor fields coincide if and only if the vector field X is a Killing field, i.e. if its flow consists of isometries. The metric Lie derivative on spinor fields coincides with another construction of the Lie derivative due to Y. Kosmann (cf. [14]).

We begin by extending the previously introduced algebraic notions to the setting of differentiable manifolds.

On a differentiable manifold M of dimension n, let $\mathscr{B}M$ denote the $\mathrm{GL}(n,\mathbb{R})$-principal bundle of frames. For any Riemannian metric g on M, the g-orthonormal frames form an $\mathrm{O}(n)$-principal bundle which we denote by O_gM. Its fibre over a point $m \in M$ is nothing but $\mathscr{B}_{g_m}T_mM$ in the notation of Sec. 1. Let h be another Riemannian metric on M, and O_hM its $\mathrm{O}(n)$-principal bundle of orthonormal frames. We denote by b_h^g the $\mathrm{O}(n)$-equivariant bundle morphism $\mathrm{O}_gM \to \mathrm{O}_hM$ which, on the fibre over a point $m \in M$, is given by $b_{h_m}^{g_m}$.

For any representation μ of $\mathrm{O}(n)$ on a vector space E, we shall also denote by b_h^g the fibrewise isomorphism

$$b_h^g : \; E_gM = \mathrm{O}_gM \times_\mu E \xrightarrow{\sim} E_hM = \mathrm{O}_hM \times_\mu E$$

defined as follows: to any element $\xi \in E_gM$, represented by an element $u \in E$ relative to a g-orthonormal frame f, we associate the element of E_hM with the same component u relative to the h-orthonormal frame $b_h^g(f)$, in other words

$$b_h^g([f,u]) = [b_h^g(f), u].$$

The equivariance of $b_h^g : \mathrm{O}_gM \to \mathrm{O}_hM$ implies that $b_h^g : E_gM \to E_hM$ is well-defined.

2.1 Remark. Whenever the $\mathrm{O}(n)$-representation μ is the restriction of a linear representation of $\mathrm{GL}(n,\mathbb{R})$, likewise denoted by μ, the two vector bundles E_gM and E_hM are both naturally identified with the *tensor bundle* $EM = \mathscr{B}M \times_\mu E$. In light of this identification, however, b_h^g, viewed as an endomorphism of EM, *does not generally coincide with the identity.* In particular, if μ is the standard representation μ_0 of $\mathrm{O}(n)$ on $\mathbb{R}^n$, the fibre bundles $\mathrm{O}_gM \times_{\mu_0} \mathbb{R}^n$ and $\mathrm{O}_hM \times_{\mu_0} \mathbb{R}^n$ are naturally identified with the tangent bundle TM and in this case b_h^g coincides with $H_g^{-1/2}$, the inverse of the positive-definite square root of the isometry $H_g = g^{-1} \cdot h : (TM, g) \to (TM, h)$.

Let us now consider the case where the manifold M is *oriented* and *spin* (in other words, we suppose that the first two Stiefel–Whitney classes $w_1(M) \in H^1(M,\mathbb{Z}_2)$ and $w_2(M) \in H^2(M,\mathbb{Z}_2)$ vanish, cf. [17]). We denote by $\widetilde{\mathrm{GL}}^+(n,\mathbb{R})$ the universal cover of the positive linear group $\mathrm{GL}^+(n,\mathbb{R})$.

We choose a *spin structure* α on M (that is, a point in a certain affine space modelled on the $\mathbb{Z}_2$-vector space $H^1(M,\mathbb{Z}_2)$) and realise it as a $\widetilde{\mathrm{GL}}^+(n,\mathbb{R})$-principal bundle $\widetilde{\mathscr{B}}^+M$, which is fibrewise a non-trivial cover of the bundle $\mathscr{B}^+M$ of positively oriented frames. For any metric g, the choice of $\widetilde{\mathscr{B}}^+M$ determines a *spinorial metric* γ in the form of a $\mathrm{Spin}(n)$-principal bundle, namely the subbundle $\mathrm{Spin}_\gamma M \subset \widetilde{\mathscr{B}}^+M$ which covers the bundle SO_gM of positively oriented g-orthonormal frames.

For any (real or complex) representation σ of the group $\mathrm{Spin}(n)$ on a vector space Σ, we denote by $\Sigma_\gamma M = \mathrm{Spin}_\gamma M \times_\sigma \Sigma$ the associated vector bundle. We say that σ is a *spinorial representation* if it is the restriction of a representation of the Clifford algebra that is *unitary* in the sense that the image of any unit vector is a unitary endomorphism. We then call the corresponding Hermitian vector bundle a *spinor bundle.* By definition, any spinor bundle is thus a left-module for the Clifford algebra bundle Cl_gM.

The natural map β that we defined in the algebraic setting extends differentiably to the bundle of spinorial bases, and consequently, determines a family of homomorphisms between spinor bundles for different metrics that we shall again denote by β. To be precise, if γ and η are two spinorial metrics *corresponding to the same bundle* $\widetilde{\mathscr{B}}^+M$, then *to any* γ*-spinor* ψ *represented by* v *in a spinorial basis* ϕ *we associate as before the* η*-spinor* $\beta^\gamma_\eta(\psi)$ *with the same representative* v, in other words

$$\beta^\gamma_\eta([\phi, v]) = [\beta^\gamma_\eta(\phi), v].$$

Again, this correspondence is well-defined because of the $\mathrm{Spin}(n)$-equivariance of the natural map β^γ_η. If σ is a spinorial representation, then β^γ_η is compatible with Clifford multiplication in the sense that for any exterior form ω, viewed as an element of $\mathrm{Cl}_g M$, and any γ-spinor ψ, one has

$$\beta^\gamma_\eta(\omega \cdot_\gamma \psi) = b^g_h(\omega) \cdot_\eta \beta^\gamma_\eta(\psi). \tag{1}$$

Any diffeomorphism ξ on M lifts to an $\mathrm{O}(n)$-equivariant diffeomorphism ξ^{O} of $\mathrm{O}_g M$ by setting

$$\xi^{\mathrm{O}} = b_g^{(\xi^{-1})^* g} \circ T\xi.$$

Such a lift exists also in the spinoral case to $\mathrm{Spin}_\gamma M$, provided that ξ preserves both the orientation and the spin structure α. However, this lift is a priori only defined up to the action of $\mathbb{Z}_2$ (cf. [14]). Nevertheless, it is unambiguously defined if the diffeomorphism ξ is connected to the identity by a path. We then denote this lift by ξ^{Spin}.

This allows us to formulate the following definition of the metric Lie derivative of a (local) field of orthonormal frames.

2.2 Definition. Let X be a vector field on a Riemannian manifold (M, g) whose local flow is denoted by (ξ_t). The g-Lie derivative of a field of g-orthonormal frames F with respect to X at a point $x \in M$ is the element $\mathscr{L}^g_X F$ of $\mathfrak{o}(n) = \mathrm{Lie}(\mathrm{O}(n))$ defined by

$$\mathscr{L}^g_X F = \frac{\mathrm{d}}{\mathrm{d}t}(\xi^{\mathrm{O}}_{-t} \circ F \circ \xi_t)\big|_{t=0} \tag{2}$$

via the canonical identification of the vertical tangent space at $F(x)$ with $\mathfrak{o}(n)$.

If M carries a spinorial metric γ, the γ-Lie derivative of a field of γ-spinorial frames Φ with respect to X is the element $\mathscr{L}^\gamma_X \Phi$ of $\mathfrak{spin}(n) = \mathrm{Lie}(\mathrm{Spin}(n))$ given by

$$\mathscr{L}^\gamma_X \Phi = \frac{\mathrm{d}}{\mathrm{d}t}(\xi^{\mathrm{Spin}}_{-t} \circ \Phi \circ \xi_t)\big|_{t=0}. \tag{2'}$$

Whenever the spinorial frame Φ is the lift $\widetilde{F}$ of an orthonormal frame F, the metric Lie derivative $\mathscr{L}^\gamma_X \widetilde{F}$ is the image of $\mathscr{L}^g_X F$ under the isomorphism $\mathfrak{spin}(n) \cong \mathfrak{so}(n)$ induced by the natural projection $\mathrm{Spin}(n) \to \mathrm{SO}(n)$. Under the classical identifications of the vector spaces $\mathfrak{spin}(n)$ and $\mathfrak{so}(n)$ with the space $\mathscr{A}\mathbb{R}^n$ of antisymmetric endomorphisms of $\mathbb{R}^n$, one then has

$$\mathscr{L}^\gamma_X \widetilde{F} = \frac{1}{2} \mathscr{L}^g_X F. \tag{3}$$

2.3 Proposition. *For a field of orthonormal frames* F*, the metric Lie derivative* $\mathscr{L}^g_X F$ *and the ordinary Lie derivative* $\mathscr{L}_X F$ *(viewed as an element of* $\mathfrak{gl}(n, \mathbb{R})$*) are related by the formula*

$$\mathscr{L}^g_X F = \mathrm{Alt}(\mathscr{L}_X F) \tag{4}$$

where Alt *denotes the natural projection* $\mathfrak{gl}(n, \mathbb{R}) \to \mathfrak{o}(n)$ *that to any matrix associates its antisymmetric part.*

Proof. The metric Lie derivative $\mathscr{L}^g_X F$ may be written as

$$\mathscr{L}^g_X F = TF(X) - X^{\mathrm{O}}_{F(m)}\,, \tag{5}$$

where X^{O} denotes the *metric lift* to $\mathrm{O}_g M$ of the vector field X, defined at a frame f over the point m by

$$X^{\mathrm{O}}_f = \frac{\mathrm{d}}{\mathrm{d}t}(b_g^{\xi^*_{-t}g}(T_m\xi_t \circ f))\big|_{t=0}\,.$$

In the same way, we may write

$$\mathscr{L}_X F = TF(X) - X^{\mathrm{GL}}_{F(m)}\,, \tag{5'}$$

where X^{GL} is the natural lift of X to $\mathrm{GL}(M)$, defined at f by

$$X^{\mathrm{GL}}_f = \frac{\mathrm{d}}{\mathrm{d}t}(T_m\xi_t \circ f)\big|_{t=0}\,.$$

2.4 Lemma. *At any point $f \in \mathrm{O}_g M \subset \mathrm{GL}(M)$, we have*

$$X^{\mathrm{GL}}_f - X^{\mathrm{O}}_f = \frac{1}{2}(\mathscr{L}_X g)_f$$

where $(\mathscr{L}_X g)_f$ is the (symmetric) element of $\mathfrak{gl}(n,\mathbb{R})$ that is determined by $\mathscr{L}_X g$ via the frame f.

Proof. From the definitions of the lifts X^{O} and X^{GL} it follows that

$$X^{\mathrm{O}}_f - X^{\mathrm{GL}}_f = \left(\frac{\mathrm{d}}{\mathrm{d}t} b_g^{\xi^*_{-t}g}\big|_{t=0}\right)_f$$

using the notation of Lemma 2.4.

The result now follows directly from Remark 2.1. □

Proof of Proposition 2.3 – Continued. Proposition 2.3 is a direct consequence of Equations (5) and (5'), of Lemma 2.4 and the fact that $-\frac{1}{2}(\mathscr{L}_X g)_F$ is the symmetric part of $\mathscr{L}_X F$, as can be seen directly using that F is an orthonormal basis at every point.[2] □

2.5 Proposition. *For any function $a \in C^\infty(M, \mathrm{O}(n))$, or respectively $\alpha \in C^\infty(M, \mathrm{Spin}(n))$, we have*

$$\mathscr{L}^g_X(F \cdot a) = (\mathrm{Ad}\, a^{-1})(\mathscr{L}^g_X F) + a^{-1}(\mathscr{L}_X a), \tag{6}$$
$$\mathscr{L}^\gamma_X(\Phi \cdot \alpha) = (\mathrm{Ad}\, \alpha^{-1})(\mathscr{L}^g_X F) + \alpha^{-1}(\mathscr{L}_X a). \tag{6'}$$

Proof. Equation (6) is an immediate consequence of Proposition 2.3 and the analogous formula for the ordinary Lie derivative.

Equation (6') then follows from Equation (3). □

Having defined the metric Lie derivative on frame fields, we define the metric Lie derivative of tensor and spinor fields in the following natural way.

[2] The original article contains a sign error here: it claims that the symmetric part of $\mathscr{L}_X F$ is $\frac{1}{2}(\mathscr{L}_X g)_F$.

2.6 Definition. Let $E_\mu M$ be the vector bundle associated, via $\mathrm{O}_g M$, to a representation μ of $\mathrm{O}(n)$. For any vector field X, the metric Lie derivative $\mathscr{L}^g_X S$ of a section S of $E_\mu M$ along X is defined by

$$\mathscr{L}^g_X S = [F, \mathscr{L}_X u + T\mu(\mathscr{L}^g_X F)u]\,, \tag{7}$$

where $u \in E$ represents S relative to a local section F of $\mathrm{O}_g M$.

Let $\Sigma_\gamma M$ be the vector bundle associated, via $\mathrm{Spin}_\gamma M$, to a representation σ of $\mathrm{Spin}(n)$. The metric Lie derivative $\mathscr{L}^\gamma_X \psi$ of a section ψ of $\Sigma_\gamma M$ in the direction X is defined by

$$\mathscr{L}^\gamma_X \psi = [\Phi, \mathscr{L}_X v + T\sigma(\mathscr{L}^\gamma_X \Phi)v]\,, \tag{7'}$$

where $v \in \Sigma$ represents ψ relative to a local section Φ of $\mathrm{Spin}_\gamma M$.

Note. In the preceding formulas, $T\mu$ and $T\sigma$ denote the derivatives at the identity of the representations μ and σ.

2.7 Remark. (i) It follows immediately from Proposition 2.5 that Definition 2.6 does not depend on the choice of local frames.

(ii) If the manifold is spin, every tensor bundle can be constructed from the bundle of spinorial frames (although it is not a spinor bundle!), and the two definitions of metric Lie derivative coincide.

(iii) Like the ordinary Lie derivative, the metric Lie derivative is a derivation with respect to the tensor product on tensors and coincides with the usual derivative on scalars. As an immediate result, the metric Lie derivative is also a derivation for the tensor product on spinors and/or tensors.

2.8 Proposition. *The metric Lie derivative $\mathscr{L}^g_X$ acting on a field of μ-tensors S differs from the ordinary Lie derivative by the formula*

$$\mathscr{L}^g_X S = \mathscr{L}_X S + \frac{1}{2}((\mathscr{L}_X g)^\sharp)_{*\mu} S, \tag{8}$$

where $(\mathscr{L}_X g)^\sharp$ denotes the endomorphism associated to $\mathscr{L}_X g$ via the metric g, and $_{*\mu}$ the infinitesimal action of an endomorphism field on a μ-tensor field by the representation μ.

In consequence, the two Lie derivative coincide whenever X is a Killing vector field.

Proof. This is a consequence of Proposition 2.3 taking into account Lemma 2.4. □

The essential properties of the metric Lie derivative are collected in the following propositions.

2.9 Proposition. *The metric Lie derivative $\mathscr{L}^g$ preserves the metric g, i.e. for any vector field X,*

$$\mathscr{L}^g_X g = 0.$$

More generally, for any (orthogonal or unitary) representation μ of $\mathrm{O}(n)$ and any two μ-tensor fields S_1 and S_2, we have for any vector field X

$$\mathscr{L}_X (S_1, S_2)_g = (\mathscr{L}^g_X S_1, S_2)_g + (S_1, \mathscr{L}^g_X S_2)_g, \tag{9}$$

where $(\cdot,\cdot)_g$ *denotes the Euclidean (Hermitian) inner product determined by* μ.

If M *carries a spinorial metric* γ *and* $\Sigma_\gamma M$ *is the vector bundle associated to a unitary representation* σ *of* $\mathrm{Spin}(n)$*, then for any two section* ψ_1 *and* ψ_2 *of* $\Sigma_\gamma M$*, we have*

$$\mathscr{L}_X(\psi_1,\psi_2)_\gamma = (\mathscr{L}^\gamma_X\psi_1,\psi_2)_\gamma + (\psi_1,\mathscr{L}^\gamma_X\psi_2)_\gamma\,, \tag{9'}$$

where $(\cdot,\cdot)_\gamma$ *denotes the Euclidean (Hermitian) inner product determined by* σ *on* $\Sigma_\gamma M$*. If* σ *is a spinorial representation, then for any differential form* ω *(viewed as a section of* $\mathrm{Cl}_g M$ *acting on* $\Sigma_\gamma M$ *via the representation* σ*), we have*

$$\mathscr{L}^\gamma_X(\omega\cdot_\gamma\psi) = (\mathscr{L}^g_X\omega)\cdot_\gamma\psi + \omega\cdot_\gamma(\mathscr{L}^\gamma_X\psi), \tag{9''}$$

where $\cdot_\gamma$ *denotes the action of* $\mathrm{Cl}_g M$ *on the spinor bundle* $\Sigma_\gamma M$ *(the representation* σ *is implied).*

Proof. Equations 9 and 9' are immediate consequences of the defining Equations 7 and 7'. The vanishing of $\mathscr{L}^g_X g$ follows then from considering the standard representation of $\mathrm{O}(n)$ on $\mathbb{R}^n$.

Equation 9" can be shown as follows: by the definition of a spinorial representation, σ is the restriction to $\mathrm{Spin}(n)$ of a linear homomorphism $\mathrm{Cl}(n)\to\mathrm{End}\,\Sigma$ (again denoted by σ). Relative to a γ-spinorial frame $\widetilde{F}$ which is the lift of a g-orthonormal frame F, the Clifford multiplication of a γ-spinor field ψ by a differential form ω is written as

$$\omega\cdot\psi = [\widetilde{F},\sigma(\alpha)v]$$

where α represents the form ω in $\mathrm{Cl}(n)$ relative to F, and v represents ψ relative to $\widetilde{F}$. We thus have, using the defining Equation 7',

$$\begin{aligned}
\mathscr{L}^\gamma_X(\omega\cdot\psi) &= [\widetilde{F},\mathscr{L}_X(\sigma(\alpha)v) + T\sigma(\mathscr{L}^\gamma_X\widetilde{F})(\sigma(\alpha)v)]\\
&= [\widetilde{F},\sigma(\mathscr{L}_X\alpha)v + \sigma(\alpha)\mathscr{L}_X v + \sigma(\mathrm{ad}(T\sigma(\mathscr{L}^\gamma_X\widetilde{F}))\alpha)v + \sigma(\alpha)T\sigma(\mathscr{L}^\gamma_X\widetilde{F})v]\\
&= [\widetilde{F},\sigma(\mathscr{L}_X\alpha + \mathrm{ad}(T\sigma(\mathscr{L}^\gamma_X\widetilde{F}))\alpha)v + [\widetilde{F},\sigma(\alpha)(\mathscr{L}_X v + T\sigma(\mathscr{L}^\gamma_X\widetilde{F})v)]\\
&= (\mathscr{L}^g_X\omega)\cdot\psi + \omega\cdot\mathscr{L}^\gamma_X\psi\,.
\end{aligned}$$

□

2.10 Proposition. *On a Riemannian manifold* (M,g)*, let* a *be a function,* X *a vector field and* S *a* μ*-tensor field. Then*

$$\mathscr{L}^g_{aX}S = a\mathscr{L}^g_X S - \frac{1}{2}(\mathrm{d}a\wedge X^\flat)_{*\mu}S\,. \tag{10}$$

If M *carries a spinorial metric* γ *and* $\Sigma_\gamma M$ *is the vector bundle associated to a representation* σ *of* $\mathrm{Spin}(n)$*, then, for any section* ψ *of* $\Sigma_\gamma M$*, we have*

$$\mathscr{L}^\gamma_{aX}\psi = a\mathscr{L}^\gamma_X\psi - \frac{1}{4}(\mathrm{d}a\wedge X^\flat)\cdot_\gamma\psi\,. \tag{10'}$$

Proof. This proposition is a direct consequence of the defining Equation 7 in view of Proposition 2.3 and, in the spinorial case, of Equation (3). □

We may compare the metric Lie derivative to the Levi-Civita connection. In order to do so, we recall that the Levi-Civita connection ∇^g associated to the metric g lifts to the bundle of spinorial frames. We naturally denote the thus induced connection by ∇^γ. This gives us a covariant derivative for γ-spinor fields. We must, however, treat with caution the fact that the action of the exterior algebra by Clifford multiplication yields a factor $\frac{1}{2}$ in the action of connection forms (see (3)). We thus obtain

2.11 Proposition. *The metric Lie derivative and the Levi-Civita covariant derivative are, for vector fields X and μ-tensors fields S, related by the formula*

$$\mathscr{L}^g_X S = \nabla^g_X S - \frac{1}{2}(\mathrm{d}X^\flat)_{*\mu} S \tag{11}$$

(where $X^\flat$ denotes the 1*-form dual to X via the metric g, and* d *the exterior derivative), and for a γ-spinor field ψ by the formula*

$$\mathscr{L}^\gamma_X \psi = \nabla^\gamma_X \psi - \frac{1}{4}(\mathrm{d}X^\flat)\cdot_\gamma \psi\,. \tag{11'}$$

Proof. Equations (11) and (11') are easily verified using the defining Equation (7) together with the analogous formula for the covariant derivative. They may also be deduced from Proposition 2.10 with the following universal argument: by Proposition 2.10, the mapping

$$a \mapsto \mathscr{L}^g_{aX} + \frac{1}{2}(\mathrm{d}a \wedge X^\flat)_{*\mu}$$

is a differential operator of order 0 in a. The same applies to

$$a \mapsto \nabla^g_{aX}.$$

The difference between the two sides of (11) is thus tensorial in X. It then suffices to note that this 1-form is invariant under g-orthogonal transformations, hence it must be zero. □

2.12 Proposition. *The curvature of the metric Lie derivative $\mathscr{L}^g$, i.e. its failure to be a Lie algebra homomorphism between the Lie algebra $\mathscr{T}M$ of vector fields on M and the Lie algebra of differential operators on vector fields, is given by the formula*

$$[\mathscr{L}^g_X, \mathscr{L}^g_Y] - \mathscr{L}^g_{[X,Y]} = -\frac{1}{4}[(\mathscr{L}_X g)^\sharp, (\mathscr{L}_Y g)^\sharp]. \tag{12}$$

Proof. This is verified directly, starting from Proposition 2.3 and using the fact that the antisymmetric part of the commutator of two endomorphisms is the sum of the commutator of the antisymmetric parts and the commutator of the symmetric parts. □

2.13 Remark. The right hand side of Equation (12) vanishes as soon as X (or Y) is an infinitesimal conformal transformation, since then $(\mathscr{L}_X g)^\sharp$ (or $(\mathscr{L}_Y g)^\sharp$) is a multiple of the identity. Because of this, one may consider that calling $\mathscr{L}^g_X$ a "Lie derivative" is only really justified for these kinds of vector fields, which is in accord with the remark that one can find on page 101 of [19].

3 Variations of the Dirac operator and its eigenvalues under changes of the metric

Recall that for any γ-spinor field ψ (relative to any spinorial representation σ),

$$\mathscr{D}^\gamma \psi = \sum_{i=1}^n f_i \cdot_\gamma \nabla^\gamma_{f_i} \psi,$$

where (f_i) is a g-orthonormal basis at the point in question.

Now that we know how to compare spinor fields for two spinorial metrics γ and η (in the same spinorial class, but belonging to two distinct metrics g and h), it becomes possible to compare the Dirac operators $\mathscr{D}^\gamma$ and $\mathscr{D}^\eta$ acting on γ- and η-spinor fields, respectively. We shall do this by introducing the transported operator

$$^\gamma\mathscr{D}^\eta = (\beta^\gamma_\eta)^{-1} \circ \mathscr{D}^\eta \circ \beta^\gamma_\eta. \tag{13}$$

3.1 Theorem. *The transport $^\gamma\mathscr{D}^\eta$ of the η-Dirac operator is, on any γ-spinor field ψ, expressed as*

$$\begin{aligned} ^\gamma\mathscr{D}^\eta \psi = &\sum_{i=1}^n f_i \cdot_\gamma \nabla^\gamma_{H_g^{-1/2}(f_i)} \psi \\ &+ \frac{1}{2} \sum_{i=1}^n f_i \cdot_\gamma H_g^{1/2} (\nabla^g_{H_g^{-1/2}(f_i)} H_g^{-1/2} + {}^gA^h_{H_g^{-1/2}(f_i)} \circ H_g^{-1/2}) \cdot_\gamma \psi, \end{aligned} \tag{14}$$

where (f_i) is a g-orthonormal frame and ${}^gA^h = \nabla^h - \nabla^g$ is the difference of the two Levi-Civita connections for h and g, respectively.

Proof. By the definition of the transported Dirac operator, we have

$$^\gamma\mathscr{D}^\eta \psi = \sum_{i=1}^n f_i \cdot_\gamma ({}^\gamma\nabla^\eta_{H_g^{-1/2}(f_i)} \psi),$$

where ${}^\gamma\nabla^\eta = (\beta^\gamma_\eta)^{-1} \circ \nabla^\eta \circ \beta^\gamma_\eta$ acting on sections of $\mathrm{Spin}_\gamma M$ is the image of the Levi-Civita connection ∇^η under the natural map β^γ_η. It is in fact induced by the connection ${}^g\nabla^h = H_g^{1/2} \circ \nabla^h \circ H_g^{-1/2}$ defined on $\mathrm{SO}_g M$, which is the transport of ∇^h under the map b^g_h.

Equation (14) now follows from a direct calculation using Equation (1) and the fact that $(b^g_h(f_i))$ is an h-orthonormal basis (recall that $b^g_h(f_i) = H_g^{-1/2}(f_i)$). □

3.2 Remark. (i) Observing that the principal symbol of $^\gamma\mathscr{D}^\eta$ is given by the first term on the right hand side of (14), we would like to point out that this operator is itself *not* a Dirac operator.

(ii) The other term on the right hand side of (14) involves the expression

$$H^{1/2}(\nabla^g H^{-1/2} + {}^gA^h \circ H^{-1/2}),$$

which is precisely the difference ${}^g\nabla^h - \nabla^g$.

Since the connection ${}^g\nabla^h$ is metric, $H^{1/2}(\nabla^g_X H^{-1/2} + {}^gA^h_X \circ H^{-1/2})$ is antisymmetric for any X and may thus be considered as an element of the Clifford algebra.

In order to study the variation of $\mathscr{D}^\eta$ when h varies in the space of metrics $\mathscr{M}M$, we replace $\mathscr{D}^\eta$ with the transported operator ${}^\gamma\mathscr{D}^\eta$ which acts on sections of the (fixed!) bundle of γ-spinors.

The tangent vectors to a metric $g \in \mathscr{M}M$ will be identified with symmetric bilinear forms. For example, a symmetric bilinear form k is the tangent vector at $t = 0$ to the curve of metrics $t \mapsto g_t = g + tk$.

3.3 Theorem. *The infinitesimal variation of the Dirac operator at a spinoral metric γ in the direction of changing the Riemannian metric by k is given by the formula*

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}{}^\gamma\mathscr{D}^{\gamma_t}\big|_{t=0}\right)\psi = -\frac{1}{2}\sum_{i=1}^{n} f_i \cdot \gamma\nabla^\gamma_{K_g(f_i)}\psi + \frac{1}{4}(\delta^g k + \mathrm{d}(\mathrm{tr}_g\, k)) \cdot_\gamma \psi\ , \tag{15}$$

where (f_i) is a g-orthonormal frame and where δ^g denotes the divergence operator acting on symmetric 2-tensor fields.

Proof. By Theorem 3.1, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^\gamma\mathscr{D}^{\gamma_t}\big|_{t=0} = -\frac{1}{2}\sum_{i=1}^{n} f_i \cdot_\gamma \nabla^g_{K_g(f_i)} + \sum_{i=1}^{n} f_i \cdot_\gamma \left(\nabla^g_{f_i}\frac{\mathrm{d}}{\mathrm{d}t}(G_t)_g^{-1/2}\big|_{t=0} + \frac{\mathrm{d}}{\mathrm{d}t}{}^g A^{g_t}_{f_i}\big|_{t=0}\right). \tag{16}$$

The 3-tensors occurring in the second term are respectively equal to

$$\nabla^g\frac{\mathrm{d}}{\mathrm{d}t}(G_t)_g^{-1/2}\big|_{t=0}(X,Y,Z) = -\frac{1}{2}(\nabla^g_X k)(Y,Z)$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}{}^g A^{g_t}\big|_{t=0}(X,Y,Z) = \frac{1}{2}((\nabla^g_X k)(Y,Z) + (\nabla^g_Y k)(X,Z) - (\nabla^g_Z k)(X,Y))\ .$$

These quantities are symmetric in the last two and the first two entries, respectively. As a consequence, the 3-form terms in the Clifford product formula do not contribute, leaving only the contractions, which yield the second summand in (15). □

3.4 Remark. The space $\mathscr{M}M$ of Riemannian metrics on M has an interesting geometry; it carries in particular an action of the diffeomorphism group $\mathrm{Diff}(M)$ of M. Varying the Riemannian metric by action of diffeomorphisms does not change the geometry. Since we only care about those infinitesimal variations of the metric that *change* the geometry, it is helpful to restrict to variations that are transverse to this action. This is ensured by imposing that k satisfies the condition $\delta k = 0$. (In this way, its inner product with respect to the standard Riemannian metric on $\mathscr{M}M$ with any *trivial* infinitesimal variation $\mathscr{L}_X g$ is then zero; this is because the divergence δ is, up to a factor $\frac{1}{2}$, the adjoint to the operator $X \mapsto \mathscr{L}_X g$.) For such a variation, Equation (15) simplifies by vanishing of the δk term.

We now turn to the *variation of the eigenvalues of the Dirac operator under a change of the metric.* Throughout the following, we assume that we are dealing with a spinorial metric γ whose corresponding Riemannian metric is subject to an infinitesimal variation k, to which we associate the variation $g_t = g + tk$.

We recall that, according to a result of Rellich (cf. [13, Thm. 3.9], see also [13, Lem. 3.15]), the *analytic* family of operators $\mathscr{D}^{\gamma_t}$ admits an analytic spectral decomposition $(\lambda_t^{(i)}, \Pi_t^{(i)})$ where, for each t, λ_t^i is an eigenvalue and Π_t^i the corresponding spectral projector. Let λ

be a fixed eigenvalue of $\mathscr{D}^\gamma$, then there exist m branches $\lambda_t^{(1)}, \ldots, \lambda_t^{(m)}$ passing through λ, i.e. such that $\lambda_0^{(1)} = \ldots = \lambda_0^{(m)} = \lambda$. Hence any infinitesimal variation k of the metric determines m "first derivatives" $\mathrm{d}\lambda^{(r)}/\mathrm{d}t\big|_{t=0}$ for $1 \le r \le m$ (where $m = 1$ if the eigenvalue is simple or if its multiplicity is constant in the direction of k). We denote by $E_\lambda^{(r)}$ the corresponding eigenspaces.

We thus arrive at the following result.

3.5 Theorem. *For any $r = 1, \ldots, m$, the first derivative of λ corresponding to the branch $\lambda_t^{(r)}$ in the direction of k may be written as*

$$\frac{\mathrm{d}\lambda_t^{(r)}}{\mathrm{d}t}\Big|_{t=0} = -\frac{1}{2}\int_M (k, Q_{\psi^{(r)}})\,\mathrm{vol}_g, \tag{17}$$

where $\psi^{(r)}$ is any unit element of the eigenspace $E_\lambda^{(r)}$, and where, for any spinor field ψ, Q_ψ is the real symmetric bilinear form determined by

$$Q_\psi(X,Y) = \frac{1}{2}\,\mathrm{Re}((X \cdot_\gamma \nabla_Y^\gamma \psi, \psi) + (Y \cdot_\gamma \nabla_X^\gamma \psi, \psi)). \tag{17'}$$

Proof. Due to the theorem of Rellich cited above, $\psi^{(r)}$ extends to an analytic family $\psi_t^{(r)}$ of unit spinors with respect to the L^2 norm that are eigenspinors of $\mathscr{D}^{\gamma_t}$ for the eigenvalue $\lambda_t^{(r)}$.

We thus have

$$\frac{\mathrm{d}\lambda_t^{(r)}}{\mathrm{d}t}\Big|_{t=0} = \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_M (\psi_t^{(r)}, {}^\gamma\mathscr{D}^{\gamma_t}\psi_t^{(r)})\,\mathrm{vol}_g\right)\Big|_{t=0}$$

which may be expanded into three terms. The two terms that contain a derivative of the eigenvector give zero contribution since they involve the L^2 inner product of $\psi^{(r)}$ with $\mathrm{d}\psi^{(r)}/\mathrm{d}t$, which vanishes because of the normalisation condition on $\psi_t^{(r)}$. This leaves only the term that includes the derivative of the Dirac operator. We may thus use Equation (15) which yields the asserted formula (17) when we take into account that the term $\delta^g k + \mathrm{d}\,\mathrm{tr}_g\, k$ also does not contribute since its Clifford product action on $\psi^{(r)}$ is antisymmetric. □

3.6 Remark. The quadratic form $-\frac{1}{2}Q_{\psi^{(r)}}$ may be considered as the r-th branch of the gradient of λ with respect to the standard inner product on $\mathscr{M}M$.

One may also adopt another point of view and consider the map $\psi \mapsto -\frac{1}{2}Q_\psi$ as a quadratic form defined on the eigenspace E_λ with values in the space of symmetric 2-tensors on M. Its trace (with respect to the L^2 metric on E_λ) can thus be interpreted as the gradient of the function $\eta \mapsto \mathrm{Trace}({}^\gamma\mathscr{D}^\eta)\big|_{\mathrm{im}\,\Pi}$, where Π is the spectral projector onto the sum of the eigenspaces that result from deforming E_λ in a neighbourhood of the metric γ. This function is nothing but the sum of eigenvalues that emanate from λ in this deformation.

The quadratic form Q_ψ associated to an eigenspinor ψ for a nonzero eigenvalue λ can never be zero since its trace with respect to g is $\lambda\|\psi\|^2$. In the case of a non-zero eigenvalue, we shall limit ourselves to variations of the metric that fix the total volume, i.e. to infinitesimal variations k such that $\int_M(\mathrm{Trace}_g k)\,\mathrm{vol}_g = 0$. We say that a nonzero eigenvalue λ (perhaps with multiplicity) is *critical* if its gradient in the sense of Remark 3.6, i.e. the trace of the form Q taken over E_λ, is a multiple of the metric g (which is automatically zero if $\lambda = 0$).

Traditionally, the eigenspinors for the eigenvalue 0 are called *harmonic spinors*. In the case where the dimension n of M is even, the space of harmonic spinors splits as an orthogonal direct sum into harmonic spinors of positive and negative chirality. Since the Clifford multiplication by a vector exchanges chiralities, it follows immediately from (17') that the restriction of Q to each of the two subspaces vanishes. In particular, the trace of Q is zero on the space of harmonic spinors, that is, 0 is a critical eigenvalue for any metric (provided there is at least one non-trivial harmonic spinor).

In the case where all nontrivial harmonic spinors have the same chirality (which can only occur non trivially if n is a multiple of 4, due to the Index Theorem), the quadratic form Q is identically zero on the space of harmonic spinors. The dimension of the kernel is thus minimal (again by the Index Theorem); it cannot decrease in a neighbourhood of the considered point in the space of metrics.

One question naturally arises: are metrics whose harmonic spinors have only one chirality generic? To our knowledge, this question is still open.

An eigenvalue of the Dirac operator does not change when the metric is varied by action of diffeomorphisms (which preserve the spinorial structure). As a result, the differential of the eigenvalue in the space $\mathscr{M}M$ of Riemannian metrics vanishes on the tangent space at g to the orbit under the action of diffeomorphisms. This implies that the divergence of the quadratic form Q_ψ must be zero for any eigenspinor field ψ (cf. [4, Chap. 4]). One may verify this fact directly by the following calculation (we omit the indices g and γ):

$$\begin{aligned}
(\delta Q_\psi)(X) &= -\sum_{i=1}^{n} (\nabla_{f_i} Q_\psi)(f_i, X) \\
&= \frac{1}{2} \operatorname{Re} \left(\sum_{i=1}^{n} ((f_i \cdot \nabla^2_{f_i,X}\psi, \psi) + (X \cdot \nabla_{f_i}\psi, \nabla_{f_i}\psi) + (\nabla_X \psi, \mathscr{D}\psi) - (X \cdot \mathscr{D}\psi, \psi) \right) \\
&= \frac{1}{2} \operatorname{Re} \left(-(\nabla_X \mathscr{D}\psi, \psi) + (\nabla_X \psi, \mathscr{D}\psi) - (X \cdot \mathscr{D}\psi, \psi) \right)
\end{aligned}$$

and the last expression clearly vanishes when ψ is an eigenspinor.

Let us examine the particular case where the change of metric is *conformal*. If $h = e^{2u} g$, then Equation (14) reduces to

$$ {}^{\gamma}\mathscr{D}^{\eta}\psi = e^{-u} \left(\mathscr{D}^{\gamma}\psi + \frac{n-1}{2} \mathrm{d}u \cdot_\gamma \psi \right), \tag{18}$$

or alternatively

$$ {}^{\gamma}\mathscr{D}^{\eta} = e^{\frac{n+1}{2}u} \circ \mathscr{D}^{\gamma} \circ e^{-\frac{n-1}{2}u}. \tag{18'}$$

This fact reflects the existence of a *conformal Dirac operator* defined on sections of the spinor bundle with conformal weight $-\frac{1}{2}(n-1)$, both of which may be defined purely in terms of the conformal structure (cf. [12] where one must take care that TM is taken with conformal weight -1, while in our case the tangent bundle has conformal weight $+1$.)

The derivative of an eigenvalue with respect to the variation $k = 2ag$ is by Theorem 3.5 equal to

$$\frac{\mathrm{d}\lambda_t^{(r)}}{\mathrm{d}t}\Big|_{t=0} = -\lambda \int_M a |\psi^{(r)}|^2 \operatorname{vol}_g.$$

The Dirac eigenvalue 0 is thus always critical for conformal variations. This is a consequence of Equation (18') which furnishes an isomorphism between the spaces of harmonic spinors for two conformally related metrics.

In the case of conformal variations, we have the following result as a corollary of Equation (18').

3.7 Proposition. *A nonzero eigenvalue λ is critical for conformal variations of the metric that fix the total volume, i.e. for those infinitesimal variations $k = 2ag$ such that $\int_M a \operatorname{vol}_g = 0$, if and only if the corresponding eigenspinor has constant length.*

Recall that a γ-spinor field ψ is called a γ-*Killing spinor* if it satisfies the relation

$$\nabla^\gamma_X \psi = -\frac{\lambda}{n} X \cdot_\gamma \psi$$

for any tangent vector X and some $\lambda \in \mathbb{R}$. Such a spinor field is clearly an eigenspinor of the Dirac operator for the eigenvalue λ. One can show that this eigenvalue is in absolute value the smallest eigenvalue of $\mathscr{D}$ and that any eigenspinor for this eigenvalue is a Killing spinor (cf. [10, 11]). The existence of Killing spinors imposes quite restrictive conditions on the Riemannian metric g, in particular g is an Einstein metric with scalar curvature $\frac{4(n-1)}{n}\lambda^2$. As an immediate consequence of Equations (17) and (17'), we have the following proposition.

3.8 Proposition. *If (M, γ) carries a nontrivial Killing spinor, the corresponding eigenvalue λ is critical for variations of g among metrics having a given total volume.*

Note that since the eigenspace E_λ consists of Killing spinors, the quadratic form Q (and not only its trace) is proportional to g.

We give an example of a spin manifold where all eigenvalues of the Dirac operator are critical.

3.9 Proposition. *The Dirac eigenvalues of the standard spinorial metric on the sphere S^n are all critical (for variations that preserve the total volume).*

Proof. Let γ denote the standard spinorial metric (with constant sectional curvature equal to 1). The eigenvalues of $\mathscr{D}^\gamma$ are precisely the numbers $\pm(k + \frac{n}{2})$, where $k \in \mathbb{N}$ (cf. [2]). The eigenspinors corresponding to the minimal eigenvalues $\pm\frac{n}{2}$ are Killing spinors. The spinor bundle $\Sigma_\gamma S^n$ is trivialised by an orthonormal basis (ψ_α) of $E_{n/2}$ (or of $E_{-n/2}$). Following [2], we may introduce the operators

$$A^+ = (\mathscr{D} + \tfrac{1}{2})^2 \qquad \text{and} \qquad A^- = (\mathscr{D} - \tfrac{1}{2})^2$$

whose eigenspaces $\mathscr{A}_k^\pm$ are the tensor products $\mathscr{H}_k \otimes E_{\pm(k+n/2)}$, where $\mathscr{H}_k$ denotes the space of spherical harmonics of order k, i.e. the space of eigenfunctions of the Riemannian Laplacian for the eigenvalues $k(n + k - 1)$.

One easily establishes the isomorphisms

$$\mathscr{A}_k^+ = E_{k+n/2} \oplus E_{-(k-1+n/2)}, \qquad 1 \le k < \infty, \tag{19$^+$}$$
$$\mathscr{A}_k^- = E_{-(k+n/2)} \oplus E_{k-1+n/2}, \qquad 1 \le k < \infty. \tag{19$^-$}$$

It follows directly from (17') that the trace of the quadratic form Q restricted to the subspaces $\mathscr{A}_k^\pm$ is proportional to g. For this, it suffices to evaluate it on a basis that respects the direct sum.

The proof is finished by a recursive argument using the orthogonal decompositions (19^+) and (19^-) together with the fact that the quadratic form is a multiple of the metric g for any Killing spinor, i.e. for any element of $\mathscr{A}_0^\pm = E_{\pm n/2}$. □

3.10 Remark. One may compare this result with the result obtained in [3] for the Riemannian Laplacian acting on functions.

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